

Supersymmetric Skyrmions : Numerical Solutions

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We consider the $\mathcal{N} = 1$ Skyrme model and obtain supersymmetric skyrmion solutions numerically. The model necessarily contains higher derivative terms and as a result the field equation becomes a fourth-order differential equation. Solving the equation directly leads to runaway solutions as expected in higher derivative theories. We, therefore, apply the perturbation method and show that skyrmion solutions exist upto the second order in the coupling constant.

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1. INTRODUCTION

Soliton solutions in non- or supersymmetric gauge theories at large- N have been playing an important role to understand non-perturbative phenomena in QCD.

A well-known example is the Skyrme model and its soliton solutions, called skyrmions [1]. Witten investigated mesons and glue states in the large- N limit of QCD by a systematic expansion in powers of $1/N$ and showed that baryons emerge as solitons with mass of $O(N)$ [2]. In the successive papers [3], it was shown that the resultant effective theory is the Skyrme model and skyrmions are interpreted as baryons [4].

More recently, it was found that domain wall solutions exist in large- N SUSY gluodynamics (SQCD) [5] as well as in non-SUSY QCD [6, 7]. These walls behave as D-branes on which the string could end if they are BPS saturated [8]. An interesting observation is that the wall width is $O(1/N)$ and correspondingly heavy states are expected to emerge as solitons with mass of $O(N)$ [9].

In this context, it may be natural to consider that solitons with mass of $O(N)$ in SQCD could have something to do with skyrmions in the supersymmetric version of the Skyrme model. The supersymmetric Skyrme model was constructed in Ref. [10]. The extension to supersymmetry restricts us to work on a $CP(1)$ target space rather than S^3 . The authors concluded that the possibility of the existence of soliton solutions is not excluded when the higher derivative terms are taken into account.

In this paper we construct supersymmetric skyrmion solutions numerically. In the supersymmetric case, the Skyrme field equation is not second-order but fourth-order in derivatives. In general, higher derivative theories lack a lowest-energy state and exhibit runaway solutions along with physical solutions, which makes numerical computation unstable. The prescription was proposed by Simon in Ref. [11]. According to his argument, if higher-order derivative terms can be considered as a small perturbation, physical solutions should be Taylor-expandable around the leading order solution. We shall apply this perturbation method to the supersymmetric Skyrme field equation and obtain soliton solutions upto the second order in the coupling constant. The dependence of the skyrmion solutions on the coupling constant is also examined.

2. THE SUPERSYMMETRIC SKYRME MODEL

In this section, we give a brief review of the $\mathcal{N} = 1$ supersymmetric Skyrme model constructed in Ref. [10]. In supersymmetric theories, the target manifold \mathcal{M} must be Kähler [12]. It was shown in Ref. [10] that the only nontrivial homotopy group in four dimensional spacetime is $\pi_3(CP(1)) = Z$. The complex projective space $CP(N)$ is realised by gauging the $U(1)$ subgroup of $SU(N)$, *i.e.* $CP(N) \equiv SU(N)/SU(N-1) \times U(1)$. This amounts to replacing ordinary derivatives in the Lagrangian into covariant derivatives $D_\mu = \partial_\mu - iV_\mu U \tau_3$ where τ_i ($i = 1, 2, 3$) are Pauli matrices. The Skyrme Lagrangian is then given by

$$\mathcal{L} = -\frac{f_\pi^2}{16} \text{tr}(D^\mu U^\dagger D_\mu U) + \frac{1}{32e^2} \text{tr}[U^\dagger D_\mu U, U^\dagger D_\nu U]^2. \quad (1)$$

where f_π is a pion decay constant and e is a dimensionless constant. Let us parameterise the chiral field in terms of the complex scalars $A = (A^1, A^2) \in C^2$ with $\bar{A}A \equiv A_1^* A_1 + A_2^* A_2 = 1$. A_i is related to $SU(2)$ matrix by

$$U = \begin{pmatrix} A_1 & -A_2^* \\ A_2 & A_1^* \end{pmatrix}. \quad (2)$$

One can parameterise the gauge field in terms of A_i as

$$V_\mu = -\frac{i}{2}[\bar{A}\partial_\mu A - (\partial_\mu \bar{A})A]. \quad (3)$$

The Skyrme Lagrangian (1) is then written as

$$\mathcal{L} = -\frac{f_\pi^2}{8}\bar{D}^\mu \bar{A}D_\mu A + \frac{1}{16e^2}(B_{[\mu}^* B_{\nu]})^2 \quad (4)$$

where

$$B_\mu = i\epsilon^{ij}A_i\partial_\mu A_j. \quad (5)$$

Now, $\omega = U^\dagger \partial_\mu U dx^\mu$ is an $SU(2)$ -valued one-form and therefore the Maurer-Cartan equation holds

$$d\omega + \omega \wedge \omega = 0, \quad (6)$$

which reads

$$F_{\mu\nu} \equiv \partial_{[\mu} V_{\nu]} = -iB_{[\mu}^* B_{\nu]}. \quad (7)$$

Thus, the Lagrangian (4) becomes

$$\mathcal{L} = -\frac{f_\pi^2}{8}\bar{D}^\mu \bar{A}D_\mu A - \frac{1}{16e^2}F_{\mu\nu}^2. \quad (8)$$

To supersymmetrise the model, let us extend (A_i, V_μ) to the chiral multiplet and vector multiplet respectively

$$A_i \rightarrow (A_i, \psi_{\alpha i}, F_i), \quad V_\mu \rightarrow (V_\mu, \lambda_\alpha, D) \quad (9)$$

where $i, \alpha = 1, 2$ and F_i are complex scalars, D is a real scalar and $\psi_{\alpha i}, \lambda_\alpha$ are Majorana spinors. Then, the supersymmetric Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{SUSY} = & \frac{f_\pi^2}{8} \left[-\bar{D}^\mu \bar{A}D_\mu A + \frac{i}{2}D_\mu \psi^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\psi}^{\dot{\alpha}} - \frac{i}{2}\psi^\alpha \sigma_{\mu\alpha\dot{\alpha}} \bar{D}^\mu \bar{\psi}^{\dot{\alpha}} + \bar{F}F - i\bar{A}\lambda^\alpha \psi_\alpha + iA\bar{\lambda}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} + D(\bar{A}A - 1) \right] \\ & + \frac{1}{8e^2} \left[-\frac{1}{2}F_{\mu\nu}^2 - i\partial_\mu \lambda^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\lambda}^{\dot{\alpha}} + D^2 \right]. \end{aligned} \quad (10)$$

One can show that these are invariant under the following supersymmetric transformations:

$$\delta A_i = -\epsilon^\alpha \psi_{\alpha i} \quad (11)$$

$$\delta \psi_{\alpha i} = -i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\epsilon}^{\dot{\alpha}} D_\mu A_i + \epsilon_\alpha F_i \quad (12)$$

$$\delta F_i = -i\bar{\epsilon}^{\dot{\alpha}} \bar{\sigma}_{\dot{\alpha}\alpha}^\mu D_\mu \psi_i^\alpha - i\bar{\epsilon}_{\dot{\alpha}} A_i \bar{\lambda}^{\dot{\alpha}} \quad (13)$$

and

$$\delta V_\mu = -\frac{i}{2}(\bar{\epsilon}_{\dot{\alpha}} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \lambda_\alpha + \epsilon_\alpha \sigma_\mu^{\alpha\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}) \quad (14)$$

$$\delta \lambda_\alpha = -\epsilon^\beta \sigma_{\beta\alpha}^{\mu\nu} F_{\mu\nu} + i\epsilon_\alpha D \quad (15)$$

$$\delta D = \frac{1}{2}(\bar{\epsilon}^{\dot{\alpha}} \bar{\sigma}_{\dot{\alpha}\alpha}^\mu \partial_\mu \lambda^\alpha - \epsilon^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \bar{\lambda}^{\dot{\alpha}}). \quad (16)$$

With these transformations and the field equations, the constraints $\bar{A}A = 1$ can be extended to

$$\bar{A}A = 1, \quad \bar{A}\psi_\alpha = 0, \quad \bar{A}F = 0, \quad (17)$$

and resultantly one obtains

$$V_\mu = -\frac{i}{2}(\bar{A}\partial_\mu A - \partial_\mu \bar{A}A) + \frac{1}{2}\psi^\alpha \sigma_{\mu\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} \quad (18)$$

$$\lambda_\alpha = -i\bar{F}\psi_\alpha + \sigma_{\alpha\dot{\alpha}}^\mu \bar{\psi}^{\dot{\alpha}} D_\mu A \quad (19)$$

$$D = \bar{D}^\mu \bar{A}D_\mu A - \frac{i}{2}(D_\mu \psi^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\psi}^{\dot{\alpha}} - \psi^\alpha \sigma_{\mu\alpha\dot{\alpha}} \bar{D}^\mu \bar{\psi}^{\dot{\alpha}}). \quad (20)$$

Setting $\psi_\alpha = F_i = 0$, one arrives at the bosonic sector of Eq. (10),

$$\mathcal{L}_{SUSY} = -\frac{f_\pi^2}{8}\bar{D}^\mu\bar{A}D_\mu A + \frac{1}{8e^2}\left[-\frac{1}{2}F_{\mu\nu}^2 + (\bar{D}^\mu\bar{A}D_\mu A)^2\right]. \quad (21)$$

The last term is fourth-order in derivatives. There are other possible fourth-order terms we can add to the Lagrangian, which turn out [10]

$$\square\bar{A}\square A - (\bar{D}^\mu\bar{A}D_\mu A)^2. \quad (22)$$

The most general supersymmetric Skyrme model is thus given by

$$\mathcal{L}_{SUSY} = -\frac{f_\pi^2}{8}\bar{D}^\mu\bar{A}D_\mu A + \frac{1}{8e^2}\left[\alpha\left\{-\frac{1}{2}F_{\mu\nu}^2 + (\bar{D}^\mu\bar{A}D_\mu A)^2\right\} + \beta\left\{\square\bar{A}\square A - (\bar{D}^\mu\bar{A}D_\mu A)^2\right\}\right]. \quad (23)$$

Although we have simply set $F_i = 0$ to get Eq. (23), the Lagrangian (10) contains derivatives of F_i and hence F_i becomes a dynamical field. However, since the dynamical solution does not cancel the higher derivative terms to recover Eq. (8), the Lagrangian would be still given in the form of (23).

We consider spherically symmetric solutions with the topological charge 1. Let us impose the hedgehog ansatz

$$U = \cos f(r) + i\vec{\tau} \cdot \vec{n} \sin f(r) \quad (24)$$

where $\vec{n} = \vec{x}/r$. In terms of A , it corresponds to

$$A_1 = \cos f(r) + i \cos \theta \sin f(r), \quad A_2 = ie^{i\varphi} \sin \theta \sin f(r). \quad (25)$$

The topological charge in the Skyrme model is defined by

$$Q = -\frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{tr}(U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U) = -\frac{1}{8\pi^2} \int d^3x \epsilon^{ijk} V_i F_{jk}. \quad (26)$$

where we have used the relation (7) in the equality. Inserting the ansatz (25) into (26) and using the boundary conditions $f(0) = \pi$, $f(\infty) = 0$, one obtains $Q = 1$.

We insert the ansatz (25) into the Lagrangian (23). Then the static energy is given by

$$E = 4\pi \frac{f_\pi}{e} \int dx x^2 \left[\frac{1}{12} \left\{ (f')^2 + \frac{2 \sin^2 f}{x^2} \right\} + \frac{(\alpha + \beta)}{15} \left\{ (f')^2 - \frac{\sin^2 f}{x^2} \right\}^2 + \frac{\beta}{12} \left(f'' + \frac{2f'}{x} - \frac{\sin 2f}{x^2} \right)^2 \right]. \quad (27)$$

where we have introduced the dimensionless variable $x = ef_\pi r$ and the prime denotes a derivative with respect to x .

3. FIELD EQUATIONS

The field equation can be obtained by taking a variation with respect to $f(x)$,

$$\begin{aligned} & -x^2 f'' - 2xf' + \sin 2f + \frac{4(\alpha + \beta)}{5} \left[2f'' \sin^2 f - 6x^2 (f')^2 f'' - 4x(f')^3 + (f')^2 \sin 2f + \frac{\sin^2 f \sin 2f}{x^2} \right] \\ & + \beta \left[x^2 f^{(4)} + 4xf^{(3)} - 4f'' \cos 2f + 4(f')^2 \sin 2f - \frac{4 \sin^2 f \sin 2f}{x^2} \right] = 0. \end{aligned} \quad (28)$$

This equation contains third- and fourth-order derivative terms. In the theories which contain higher-derivative terms, there exists no lowest-energy state no matter how small their coefficients are. Thus, if one solves them directly, one would end up with picking up unphysical runaway solutions. The perturbative method to avoid this problem was proposed in Ref. [11] and has been applied for theories such as gravity and knotted solitons [13].

Let us apply the perturbation to solve the equation (28). The Skyrme model is a truncated theory of derivative expansion. Therefore, the higher-derivative terms would be treated as small corrections. Let us expand the Skyrme field in the order of the coefficient of the higher-derivative terms β ,

$$f(x) = f_0(x) + \sum_{l=1}^L \beta^l f_l(x) + O(\beta^{L+1}). \quad (29)$$

We compute solutions upto the second-order in β , *i.e.* $L = 2$.

From Eq. (28), the zeroth-order equation is given by

$$O(\beta^0) : h_0 f_0'' - 2r f_0' + \sin 2f_0 + \frac{4\alpha}{5} \left[-4r(f_0')^3 + (f_0')^2 \sin 2f_0 + \frac{\sin^2 f_0 \sin 2f_0}{r^2} \right] = 0 \quad (30)$$

where

$$h_0 = -r^2 + \frac{8\alpha}{5} (\sin^2 f_0 - 3r^2 f_0'') . \quad (31)$$

The first-order equation is given by

$$O(\beta) : h_0 f_1'' - s_1 f_1' + s_2 f_1 + s_3 + \frac{4}{5} s_4 = 0 \quad (32)$$

where

$$s_1 = 2r + \frac{8\alpha}{5} (6r^2 f_0' f_0'' + 6r f_0'^2 - f_0' \sin 2f_0) \quad (33)$$

$$s_2 = 2 \cos 2f_0 + \frac{4\alpha}{5} (2f_0'' \sin 2f_0 + 2f_0'^2 \cos 2f_0 + w_0) \quad (34)$$

$$s_3 = r^2 f_0^{(4)} + 4r f_0^{(3)} - 4f_0'' \cos 2f_0 + 4f_0'^2 \sin 2f_0 - \frac{4 \sin^2 f_0 \sin 2f_0}{r^2} \quad (35)$$

$$s_4 = 2(\sin^2 f_0 - 3r^2 f_0'') f_0'' - 4r f_0'^3 + f_0'^2 \sin 2f_0 + \frac{\sin^2 f_0 \sin 2f_0}{r^2} \quad (36)$$

and

$$w_0 = \frac{1}{r^2} (\sin^2 2f_0 + 2 \sin^2 f_0 \cos 2f_0) . \quad (37)$$

The second-order equation is given by

$$O(\beta^2) : h_0 f_2'' + \frac{4\alpha}{5} (2m_1 + m_2) + m_3 + \frac{4}{5} m_4 + m_5 = 0 \quad (38)$$

where

$$m_1 = -6r^2 f_0' f_0'' f_2' + f_0'' \sin 2f_0 f_2 + (\sin 2f_0 f_1 - 6r^2 f_0' f_1') f_1'' + f_1^2 f_0'' \cos 2f_0 - 3r^2 f_1'^2 f_0'' \quad (39)$$

$$m_2 = 2(f_0' \sin 2f_0 - 6r f_0'^2) f_2' + (2f_0'^2 \cos 2f_0 + w_0) f_2 + (\sin 2f_0 - 12r f_0') f_1'^2 + 4f_0' f_1' f_1 \cos 2f_0 \\ + \left[-2f_0'^2 \sin 2f_0 + \frac{1}{r^2} (3 \sin 2f_0 \cos 2f_0 - 2 \sin^2 f_0 \sin 2f_0) \right] f_1^2 \quad (40)$$

$$m_3 = r^2 f_1^{(4)} + 4r f_1^{(3)} - 4f_1'' \cos 2f_0 + 8f_0' f_1' \sin 2f_0 + 4(2f_0'' \sin 2f_0 + 2f_0'^2 \cos 2f_0 - w_0) f_1 \quad (41)$$

$$m_4 = 2(\sin^2 f_0 - 3r^2 f_0'') f_1'' + 2(f_0' \sin 2f_0 - 6r f_0'^2 - 6r^2 f_0' f_0'') f_1' + (2f_0'' \sin 2f_0 + 2f_0'^2 \cos 2f_0 + w_0) f_1 \quad (42)$$

$$m_5 = -2r f_2' + 2f_2 \cos 2f_0 - 2f_1^2 \sin 2f_0 . \quad (43)$$

Solutions upto the second-order are then given by

$$f(r) = f_0(x) + \beta f_1(x) + \beta^2 f_2(x) + O(\beta^3) . \quad (44)$$

4. NUMERICAL SOLUTIONS

We have solved the second-order differential equations (30,32,38) by the shooting method subject to the boundary conditions

$$f_0(0) = \pi , \quad f_0(\infty) = 0 , \quad f_1(0) = 0 , \quad f_1(\infty) = 0 , \quad f_2(0) = 0 , \quad f_2(\infty) = 0 . \quad (45)$$

Fig. 1 shows the dependence of the profile function $f(x)$ on α . The skyrmion slightly expands in size for increasing α . Fig. 2 shows its dependence on β . The difference is very small but non-zero β contributes to expand the size of

skyrmions. Fig. 3 shows the dependence on α of the first- and second-order perturbed functions. The values which give a maximum correction to the leading order are $f_1 \approx 1$ and $f_2 \approx -5$ for $\alpha = 0.8$. If we take $\beta = 0.01$, each will give a correction $O(10^{-2})$ and $O(10^{-4})$ respectively. The leading order takes the value of $O(1)$ around the maximum correction values. Thus, it seems that $\beta = 0.01$ would be small enough to make our perturbation method valid at least for $\alpha \gtrsim 0.8$. Fig. 4 shows the α dependence of the energy density. The behavior is the same as the profile, that is, the skyrmion increases in size as α increases.

The corrections gets larger as α becomes smaller, and correspondingly we have to take smaller values for β . But the regime where f_1 and f_2 take large values compared to the leading order, the derivative expansion would be broken down and the result should not be trusted.

It is noted that the first-order function contributes to increase the size of the skyrmion. On the other hand, the second-order function contributes to decrease the size. That makes the total corrections to the leading-order solution even smaller.

We have examined the dependence of the total energy on α and β which is shown in Fig. 5. For increasing α , the energy increases monotonously. For increasing β , the energy increases and the figure merely shifts upwards.

In Ref. [10], it was shown that skyrmions are unstable when $\beta = 0$. However, for $\beta \neq 0$, there is a possibility that they are stable. Unfortunately we are not able to show if our solutions are stable since the equation for the stability analysis are again higher-order in derivatives.

5. SUMMARY

In this paper we studied the $\mathcal{N} = 1$ supersymmetric Skyrme model and constructed skyrmion solutions numerically by the perturbation method upto the second order in the coefficient of the higher derivative terms. The skyrmion depends on the coupling constants α and β and increases in size when these values increase. We found that the first-order correction contributes to increasing the size of the skyrmion while the second-order contributes to decreasing the size. As a result, the total correction to the leading-order solution is very small. For $\beta = 0.01$, the first- and second-order gives a correction of 1% and 0.01% respectively to the leading order solution, which should justify our perturbative treatment of the model. The energy of the skyrmion increases monotonously for increasing α and β .

It should be also noted that for $\beta = 0$, the supersymmetric skyrmions are unstable, but the higher-derivative terms could change the stability dramatically no matter how small the correction is although we have to wait for the results from proper stability analysis.

We believe that the supersymmetric Skyrme model and its soliton solutions deserves further investigation since it may shed light on the non-perturbative effects in large- N supersymmetric QCD theory. In particular, quantisation of supersymmetric skyrmions will be necessary to be performed.

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- [1] T. H. R. Skyrme, Proc. Roy. Soc. Lond. A247 (1958) 260.
- [2] E. Witten, Nucl. Phys. B160 (1979) 57.
- [3] E. Witten, Nucl. Phys. B223 (1983) 422; Nucl. Phys. B223 (1983) 433.
- [4] G. Adkins, C. Nappi and E. Witten, Nucl. Phys. B228 (1983) 552.
- [5] G. Dvali, G. Gabadadze and Z. Kakushadze, Nucl. Phys. B562 (1999) 158.
- [6] E. Witten, Phys. Rev. Lett. 81 (1998) 2862.
- [7] M. Shifman, Phys. Rev. D59 (1999) 021501.
- [8] E. Witten, Nucl. Phys. B507 (1997) 658.
- [9] G. Gabadadze and M. Shifman, Phys. Rev. D61 (2000) 075014;
- [10] E. A. Bergshoeff, R. I. Nepomechie, H. J. Schnitzer, Nucl. Phys. B249 (1985) 93.
- [11] J. Simon, Phys. Rev. D41 (1990) 3720.
- [12] B. Zumino, Phys. Lett. B87 (1979) 203.

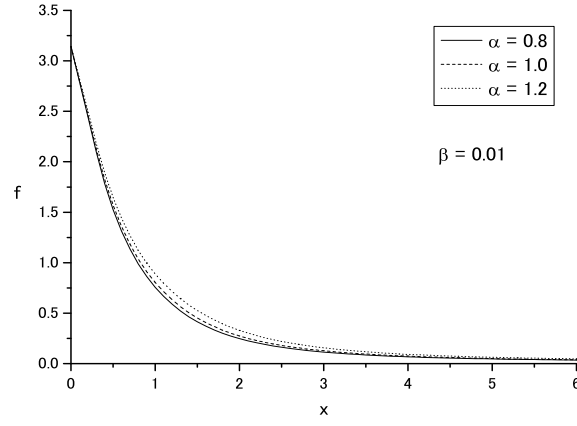


FIG. 1: The profile function f as a function of x for with $\alpha = 0.8, 1.0, 1.2$ and $\beta = 0.01$ fixed.

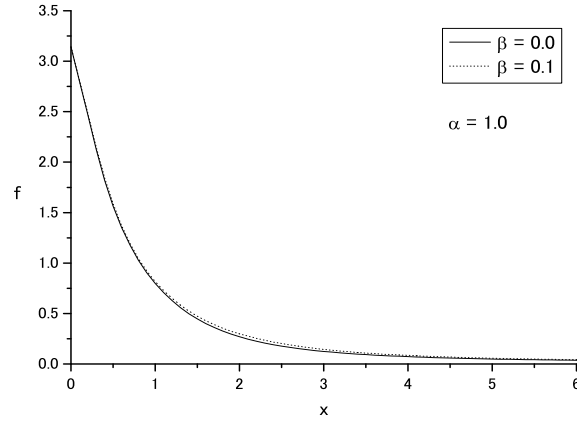


FIG. 2: The profile function f as a function of r for $\beta = 0.0, 0.1$ with $\alpha = 1.0$ fixed.

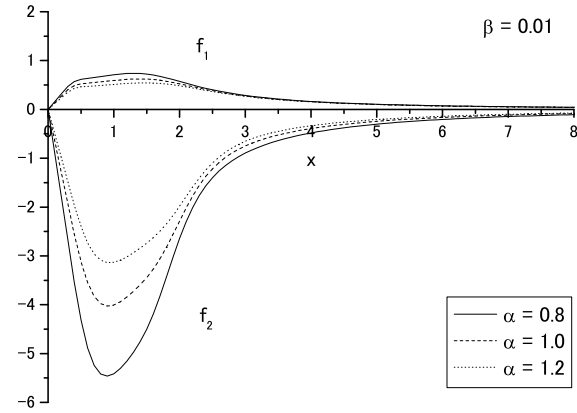


FIG. 3: The perturbed profile function f_1 and f_2 as a function of x for $\alpha = 0.8, 1.0, 1.2$.

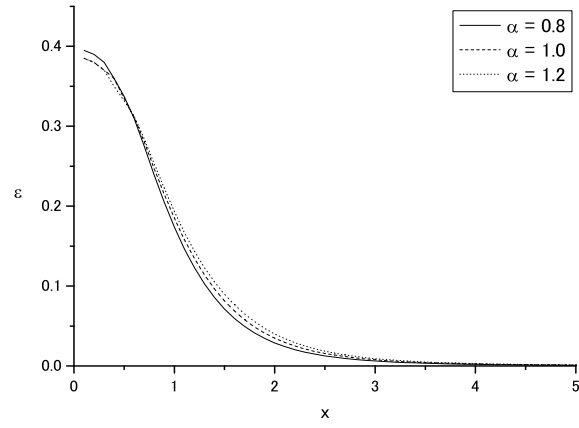


FIG. 4: The energy density as a function of x for $\alpha = 0.8, 1.0, 1.2$.

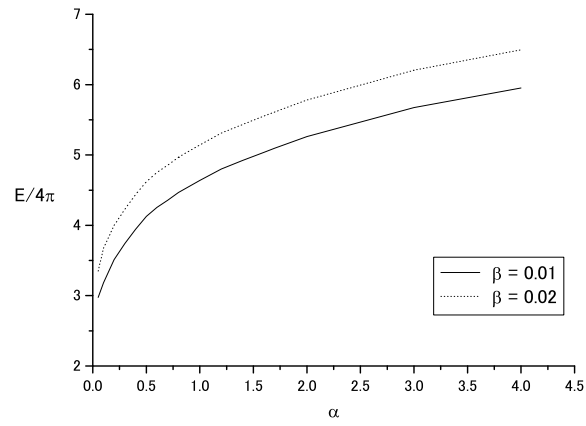


FIG. 5: The α dependence of the total energy with $\beta = 0.01$ fixed.

[13] N. Sawado, N. Shiiki and S. Tanaka, SIGMA Vol.2 (2006) 016; hep-ph/0511208; hep-ph/0507258.